A COMPARISON OF SEVERAL FINITE DIFFERENCE EQUATIONS FOR THE DISPLACEMENT OF A TAUT STRING SUBJECTED TO A TRANSVERSE LOAD

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THESIS

A Comparison of Several Finite Difference Equations for the Displacement of a Taut String Subjected to a Transverse Load

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ABSTRACT

The principle of virtual work was used to determine several typical difference equations governing the displacements of a taut string subjected to a distributed transverse load. The integrals for the strain energy and the external work were approximated by both the trapezoidal rule and Simpson's rule. Further, the integrand in the strain energy expression was replaced by two different finite-difference approximations. For comparison, two typical difference equations were developed from the differential equation for the string using central finite differences and the Hermitian differencing scheme. The finite difference equations, all of which are tridiagonal, were evaluated by comparing truncation error.

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TABLE OF SYMBOLS

CFD	first derivative in integrand evaluated by central finite difference expression
F	tensile force applied to string
h	interval between two adjacent nodes
HOD	first derivative in integrand evaluated by higher order finite difference expression
k	number of intervals
N	non-staggered scheme
q	transverse load per unit length
S	staggered scheme
U	strain energy
W	external work of transverse load
w	displacement
x	axial coordinate
δw_s	virtual displacement at sth node
τ	axial coordinate at which truncation error is evaluated
τ_s	$x_{s-1} \le \tau_s \le x_{s+1}$
$\tau_{s+\frac{1}{2}}$	$x_s \le \tau_{s+\frac{1}{2}} \le x_{s+1}$
2-4-2	order of constant coefficients from Simpson's rule, i.e., . . . $2f(x_{s-1})+4f(x_s)+2f(x_{s+1})$ . . .
4-2-4	order of constant coefficients from Simpson's rule, i.e., . . . $4f(x_{s-1})+2f(x_s)+4f(x_{s+1})$ . . .

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I. INTRODUCTION

The principle of virtual work is a very useful tool for finding the displacements of an elastic body produced by given forces [1]. In this thesis the principle is used to determine several finite difference equations governing the displacements of a taut elastic string subjected to a distributed transverse load per unit length, q. The strain energy of the string, U, is given by

$$U = \frac{F}{2} \int_{0}^{L} \left(\frac{dw}{dx} \right)^{2} dx \qquad (1.1)$$

in which F is the constant internal force in the string, w is the displacement, L is the length, and x is the axial coordinate. The positive directions of F, w, q, and x are indicated in Figure 1.

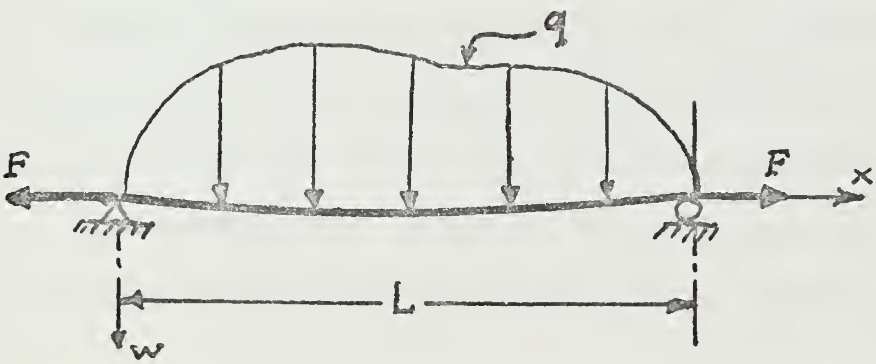

Figure 1. A STRING SUBJECTED TO A TRANVERSE LOAD AND A TENSILE FORCE

The principle of virtual work is

$$\delta U = \delta W \qquad (1.2)$$

where δU is the variation of U with respect to w and δW is the work performed by q moving through the virtual displacement δw. Hence

$$\delta W = \int_0^L q\,\delta w dx \qquad (1.3)$$

The trapezoidal rule and Simpson's rule are used to numerically evaluate the continuous integrals in Equations 1.1 and 1.3 when the string is divided into k intervals of length h. The displacements of the string are defined at the end points or nodes of these intervals, and the continuous first derivative in U is replaced by a finite difference expression. Several approximations to the first derviative are considered. This procedure produces a lengthy equation in terms of the displacements and loads at the finite number of nodes. Taking the variation of this equation with respect to the displacement at each of the nodes[1], in accordance with the principle of virtual work, leads to a set of simultaneous linear algebraic equations. There is one such set of equations for each numerical scheme. Within each set there are one or two typical difference equations. The schemes are evaluated by examining the truncation errors associated with each typical equation.

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[1]The variation must satisfy the geometric boundary conditions, i.e., $\delta w = o$ at $x = o$ and $x = L$.

A second procedure for finding the displacements of an elastic body is to use a finite difference approximation of the governing differential equation. For the string, the differential equation is

$$F \left(\frac{d^2 w}{dx^2} \right) + q = 0 \tag{1.4}$$

Substituting a finite difference approximation for the second derivative leads to a set of simultaneous linear algebraic equations of tridiagonal form. Two typical finite difference equations are derived and compared with the typical equations developed from the virtual work principle.

II. DEVELOPMENT OF THE FINITE DIFFERENCE EQUATIONS USING THE PRINCIPLE OF VIRTUAL WORK

A. THE TRAPEZOIDAL RULE AND SIMPSON'S RULE

There are many formulas available for numerically evaluating integrals. The trapezoidal rule and Simpson's rule, two of the most commonly used, were chosen for this thesis. Both of these find approximations to integrals by passing an interpolating function through several nodes and integrating the interpolating function. The trapezoidal rule uses a straight line as the interpolating function between x_s and x_{s+1}, where x_s is the value of x at the $_s$th node, and is given as [2]

$$\int_{x_s}^{x_{s+1}} f(x)dx = \frac{h}{2}\left[f(x_{s+1}) - f(x_s)\right] - \frac{h^3}{12}\, \frac{d^2 f(\tau)}{dx^2} \tag{2.1}$$

in which $x_s \leq \tau \leq x_{s+1}$. The last term in Equation 2.1 is known as the truncation error. The integral from x=0 to x=L is

$$\int_0^L f(x)dx = h\left[\tfrac{1}{2}f(0)+f(x_1)+\ldots+f(x_{k-1})+\tfrac{1}{2}f(L)\right] - \frac{h^2 L}{12}\, \frac{d^2 f(\tau)}{dx^2} \tag{2.2}$$

in which $0 \leq \tau \leq L$

In Simpson's rule a parabola is used as the interpolating function passing through the nodes x_{s-1}, x_s and x_{s+1}.

The integral from x=0 to x=L is

$$\int_0^L f(x)dx = \frac{h}{3}\left[f(0)+4f(x_1)+2f(x_2)+\ldots+2f(x_{k-2}) +4f(x_{k-1})+f(L)\right] - \frac{h^4 L}{90}\frac{d^4 f(\tau)}{dx^4} \tag{2.3}$$

where $0 \leq \tau \leq L$ and k is even.

B. DIFFERENCE APPROXIMATIONS TO THE FIRST DERIVATIVE

The equation containing the conventional central finite difference approximation to the first derivative at x_s and the truncation error is [2]

$$\left(\frac{dw}{dx}\right)_s = \frac{1}{2h}(w_{s+1} - w_{s-1}) - \frac{h^2}{6}\left(\frac{d^3 w}{dx^3}\right)_{\tau_s} \tag{2.4}$$

in which $x_{s-1} \leq \tau s \leq x_{s+1}$.

Consider the scheme in which the displacements are defined at the nodes and the first derivatives are defined at the center of the intervals as shown in Figure 2. This scheme is referred to as the staggered differencing scheme or half-station scheme. The central finite difference equation for the staggered scheme is derived in Appendix B, and is

$$\left(\frac{dw}{dx}\right)_{s+\frac{1}{2}} = \frac{1}{h}(w_{s+1} - w_s) - \frac{h^2}{48}\left(\frac{d^3 w}{dx^3}\right)_{\tau_{s+\frac{1}{2}}} \tag{2.5}$$

where $x_s \leq \tau_{s+\frac{1}{2}} \leq x_{s+1}$. Hereafter, unless noted otherwise τ_s denotes that τ is located in the interval x_{s-1} to x_{s+1} and $\tau_{s+\frac{1}{2}}$ is in the interval x_s to x_{s+1}, and τ is in the interval x=0 to x=L.

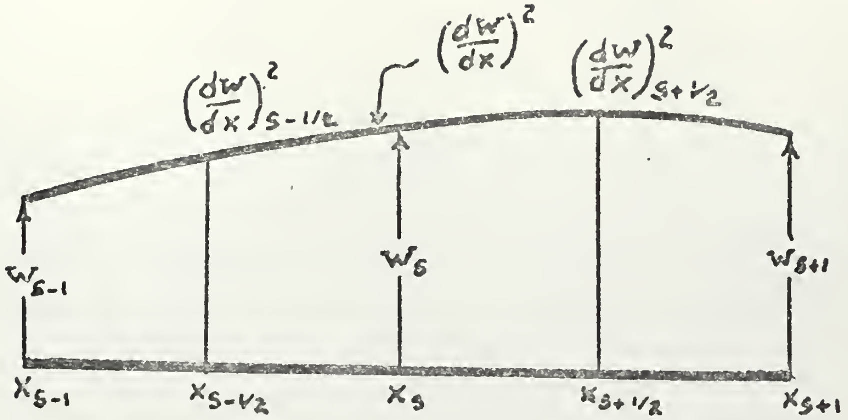

Figure 2. THE STAGGERED SCHEME

Another finite difference equation used to replace the first derivative at x_s is this higher order expression

$$\left(\frac{dw}{dx}\right)_s = \frac{1}{2h}\left(w_{s+1} - w_{s-1}\right) + \frac{h}{12F}\left(q_{s+1} - q_{s-1}\right) + \frac{7h^4}{360}\left(\frac{d^5w}{dx^5}\right)_{\tau_s} \tag{2.6}$$

The corresponding higher order staggered scheme equation is

$$\left(\frac{dw}{dx}\right)_{s+\frac{1}{2}} = \frac{1}{h}\left(w_{s+1} - w_s\right) + \frac{h}{24F}\left(q_{s+1} - q_s\right) + \frac{7h^4}{5760}\left(\frac{d^5w}{dx^5}\right)_{\tau_{s+\frac{1}{2}}} \tag{2.7}$$

Equations 2.6 and 2.7 are derived in Appendix A.

C. DEVELOPMENT OF THE TYPICAL FINITE DIFFERENCE EQUATIONS

The first typical difference equation is derived using the trapezoidal rule and the central finite difference expression. Applying Equations 2.2 and 2.4 to Equation 1.1 leads to:

$$U = \frac{Fh}{2}\left\{ \cdot \cdot \cdot + \left[\frac{1}{2h}\left(w_s - w_{s-2}\right) - \frac{h^2}{6}\left(\frac{d^3w}{dx^3}\right)_{\tau_{s-1}}\right]^2 + \left[\frac{1}{2h}\left(w_{s+1} - w_{s-1}\right) - \frac{h^2}{6}\left(\frac{d^3w}{dx^3}\right)_{\tau_s}\right]^2 + \left[\frac{1}{2h}\left(w_{s+2} - w_s\right) - \frac{h^2}{6}\left(\frac{d^3w}{dx^3}\right)_{\tau_{s+1}}\right]^2 + \cdot \cdot \cdot \right\} - \frac{h^2FL}{24} \frac{d^2\left(\frac{dw}{dx}\right)^2_{\tau}}{dx^2}$$

(2.8)

Only those terms in which w_s appears are listed in Equation 2.8 because only the typical equation is of interest. Taking the variation of Equation 2.8 with respect to the displacement w_s leads to

$$\frac{\delta U}{\delta w_s} = -\frac{F}{4h}\left(w_{s-2} - 2w_s + w_{s+2}\right) - \frac{h^2F}{12}\left[\left(\frac{d^3w}{dx^3}\right)_{\tau_{s-1}} - \left(\frac{d^3w}{dx^3}\right)_{\tau_{s+1}}\right] \qquad (2.9)$$

Taking the variation of the external work with respect to w_s leads to

$$\frac{\delta W}{\delta w_s} = hq_s \qquad (2.10)$$

Note in Equation 2.9 that the displacements are required at $_{s-2}$ and $_{s+2}$. This equation does not take into consideration the fact that the displacements at $_{s-1}$ and $_{s+1}$ are available. Hence, it appears reasonable to use these displacements instead of those at $_{s-2}$ and $_{s+2}$. Consequently, let the interval h be reduced by a factor of one half and change the subscripts accordingly to give a set of equations that do not omit any nodes. The modified equations are:

$$\frac{\delta U}{\delta w_s} = -\frac{F}{2h}\left(w_{s-1} - 2w_s + w_{s+1}\right) - \frac{h^2 F}{48}\left[\left(\frac{d^3 w}{dx^3}\right)\tau_{s-\frac{1}{2}} - \left(\frac{d^3 w}{dx^3}\right)\tau_{s+\frac{1}{2}}\right] \tag{2.11}$$

$$\frac{\delta W}{\delta w_s} = \frac{h}{2}\, q_s \tag{2.12}$$

It can be shown that equations 2.11 and 2.12 are identical to the equations developed using the trapezoidal rule and the staggered difference scheme. Using the staggered scheme eliminates the necessity to halve the internal.

As an example of the derivation of the higher order equations, a typical equation is derived using Simpson's rule in conjunction with the higher order staggered scheme. The equation for U based on Simpson's rule can be given in the form

$$U = \frac{Fh}{6}\left[\ldots + 2\left(\frac{dw}{dx}\right)^2_{s-\frac{1}{2}} + 4\left(\frac{dw}{dx}\right)^2_{s+\frac{1}{2}} + \ldots\right] - \frac{h^4 L}{90}\,\frac{d^4\left(\frac{dw}{dx}\right)^2\tau}{dx^4} \tag{2.13a}$$

Equation 2.13a could also have been given as

$$U = \frac{Fh}{6}\left[\ldots + 4\left(\frac{dw}{dx}\right)^2_{s-\frac{1}{2}} + 2\left(\frac{dw}{dx}\right)^2_{s+\frac{1}{2}} + \ldots\right] - \frac{h^4 L}{90}\,\frac{d^4\left(\frac{dw}{dx}\right)^2\tau}{dx^4} \tag{2.13b}$$

The typical equations associated with Equations 2.13a and 2.13b are different. The equation derived from Equation 2.13a is referred to as the 2-4-2 scheme and the equation derived from Equation 2.13b is referred to as the 4-2-4 scheme.

Substituting Equation 2.7 into Equation 2.13a gives

$$U = \frac{Fh}{6}\left\{ \ldots + 2\left[\frac{1}{h}(w_s - w_{s-1}) + \frac{h}{24}(q_s - q_{s-1}) + \frac{7h^4}{5760}\left(\frac{d^5w}{dx^5}\right)_{\tau_{s-\frac{1}{2}}}\right]^2 + 4\left[\frac{1}{h}(w_{s+1} - w_s) + \frac{h}{24}(q_{s+1} - q_s) + \frac{7h^4}{5760}\left(\frac{d^5w}{dx^5}\right)_{\tau_{s+\frac{1}{2}}}\right]^2 + \ldots\right\} - \frac{h^4L}{90}\,\frac{d^4\left(\frac{dw}{dx}\right)^2_{\tau}}{dx^4} \tag{2.14}$$

Taking the variation of U with respect to w_s leads to

$$\frac{\delta U}{\delta w_s} = -\frac{2F}{3h}(w_{s-1} - 3w_s + 2w_{s+1}) - \frac{h}{36}(q_{s-1} - 3q_s + 2q_{s+1}) + \frac{7h^4F}{8640}\left[\left(\frac{d^5w}{dx^5}\right)_{\tau_{s-\frac{1}{2}}} - 2\left(\frac{d^5w}{dx^5}\right)_{\tau_{s+\frac{1}{2}}}\right] \tag{2.15}$$

The variation of the external work with respect to w_s for the integral evaluated by Simpson 's rule with the 2-4-2 scheme yields

$$\frac{\delta W}{\delta w_s} = \frac{4h}{3} q_s \tag{2.16}$$

All of the typical difference equations that were derived using the principle of virtual work are given in Tables I and II. Table I contains the equations derived from the trapezoidal rule and Table II contains the equations derived from Simpson's rule.

The truncation errors listed in Tables I and II are only part of the total truncation error. These truncation errors are associated with the finite difference evaluation of the derivatives in the strain energy integral. Note that they consist of the difference of two derivatives. As a consequence, they can be converted to a higher order of h. As an example, from the first equation of Table I

$$\frac{hF}{24}\left[\left(\frac{d^3w}{dx^3}\right)_{\tau_{s-\frac{1}{2}}} - \left(\frac{d^3w}{dx^3}\right)_{\tau_{s+\frac{1}{2}}}\right] \doteq -\frac{h^2F}{24}\left(\frac{d^4w}{dx^4}\right)_{\tau_s}$$

TABLE I

TRAPEZOIDAL RULE FINITE DIFFERENCE EQUATIONS

TYPE	TYPICAL EQUATION	PARTIAL TRUNCATION ERROR
CFD	$F(w_{s-1}-2w_s+w_{s+1})/h^2 + q_s = 0$	$\frac{hF}{24}\left[\left(\frac{d^3w}{dx^3}\right)_{\tau_{s-\frac{1}{2}}} - \left(\frac{d^3w}{dx^3}\right)_{\tau_{s+\frac{1}{2}}}\right]$
HOD	$F(w_{s-1}-2w_s+w_{s+1})/h^2+\frac{1}{24}(q_{s-1}+22q_s+q_{s+1})=0$	$\frac{7h^3F}{5760}\left[\left(\frac{d^5w}{dx^5}\right)_{\tau_{s-\frac{1}{2}}} - \left(\frac{d^5w}{dx^5}\right)_{\tau_{s+\frac{1}{2}}}\right]$
CFD	Central finite difference expression used to evaluate the first derivative	
HOD	Higher order finite difference expression used to evaluate the first derivative	

TABLE II

SIMPSON'S RULE FINITE DIFFERENCE EQUATIONS

TYPE	TYPICAL EQUATION	PARTIAL TRUNCATION ERROR
CFDN 2-4-2	$F(w_{s-1}-2w_s+w_{s+1})/h^2+2q_s=0$	$\frac{hF}{24}\left[\left(\frac{d^3w}{dx^3}\right)\tau_{s-\frac{1}{2}}-\left(\frac{d^3w}{dx^3}\right)\tau_{s+\frac{1}{2}}\right]$
CFDN 4-2-4	$F(w_{s-1}-2w_s+w_{s+1})/h^2+q_s/2=0$	$\frac{hF}{24}\left[\left(\frac{d^3w}{dx^3}\right)\tau_{s-\frac{1}{2}}-\left(\frac{d^3w}{dx^3}\right)\tau_{s+\frac{1}{2}}\right]$
CFDS 2-4-2	$F(w_{s-1}-3w_s+2w_{s+1})/h^2+2q_s=0$	$\frac{hF}{24}\left[\left(\frac{d^3w}{dx^3}\right)\tau_{s-\frac{1}{2}}-2\left(\frac{d^3w}{dx^3}\right)\tau_{s+\frac{1}{2}}\right]$
CFDS 4-2-4	$F(2w_{s-1}-3w_s+w_{s+1})/h^2+q_s=0$	$\frac{hF}{24}\left[2\left(\frac{d^3w}{dx^3}\right)\tau_{s-\frac{1}{2}}-\left(\frac{d^3w}{dx^3}\right)\tau_{s+\frac{1}{2}}\right]$
HODN 2-4-2	$F(w_{s-1}-2w_s+w_{s+1})/h^2+\frac{1}{24}(q_{s-1}+46q_s+q_{s+1})=0$	$\frac{7h^3F}{5760}\left[\left(\frac{d^5w}{dx^5}\right)\tau_{s-\frac{1}{2}}-\left(\frac{d^5w}{dx^5}\right)\tau_{s+\frac{1}{2}}\right]$
HODN 4-2-4	$F(w_{s-1}-2w_s+w_{s+1})/h^2+\frac{1}{24}(q_{s-1}+10q_s+q_{s+1})=0$	$\frac{7h^3F}{5760}\left[\left(\frac{d^5w}{dx^5}\right)\tau_{s-\frac{1}{2}}-\left(\frac{d^5w}{dx^5}\right)\tau_{s+\frac{1}{2}}\right]$
HODS 2-4-2	$F(w_{s-1}-3w_s+2w_{s+1})/h^2+\frac{1}{24}(q_{s-1}+45q_s+2q_{s+1})=0$	$\frac{7h^3F}{5760}\left[\left(\frac{d^5w}{dx^5}\right)\tau_{s-\frac{1}{2}}-2\left(\frac{d^5w}{dx^5}\right)\tau_{s+\frac{1}{2}}\right]$
HODS 4-2-4	$F(2w_{s-1}-3w_s+w_{s+1})/h^2+\frac{1}{24}(2q_{s-1}+21q_s+q_{s+1})=0$	$\frac{7h^3F}{5760}\left[2\left(\frac{d^5w}{dx^5}\right)\tau_{s-\frac{1}{2}}-\left(\frac{d^5w}{dx^5}\right)\tau_{s+\frac{1}{2}}\right]$

N - non-staggered interval; S - staggered interval; 2-4-2 - Simpson's rule, 2-4-2 scheme; 4-2-4 - Simpson's rule, 4-2-4 scheme

III. THE DIFFERENTIAL EQUATION APPROACH

A. CENTRAL FINITE DIFFERENCE FORMULATION

The differential equation for the string is given by Equation 1.4. Using the conventional central finite difference expression to evaluate the second derivative at s leads to

[2]

$$F(w_{s-1} - 2w_s + w_{s+1})/h^2 + q_s - \frac{Fh^2}{12}\left(\frac{d^4w}{dx^4}\right)_{\tau_s} \qquad (3.1)$$

B. HERMITIAN FINITE DIFFERENCE FORMULATION

The Hermitian formula for the second derivative, as derived in Appendix C, is

$$(w_{s-1} - 2w_s + w_{s+1}) = \frac{h^2}{12}\left[\left(\frac{d^2w}{dx^2}\right)_{s-1} + 10\left(\frac{d^2w}{dx^2}\right)_s + \left(\frac{d^2w}{dx^2}\right)_{s+1}\right]$$

$$- \frac{h^6}{240}\left(\frac{d^6w}{dx^6}\right)_s + \ldots \qquad (3.2)$$

Applying Equation 1.4 at $_{s-1}$, $_s$, and $_{s+1}$ to Equation 3.2 gives

$$F(w_{s-1} - 2w_s + w_{s+1}) + \frac{1}{12}(q_{s-1} + 10q_s + q_{s+1})$$

$$+ \frac{h^4F}{240}\left(\frac{d^6w}{dx^6}\right)_{\tau_s} = 0 \qquad (3.3)$$

Equation 3.3 is the typical difference equation for the Hermitian finite difference scheme.

Equations 3.1 and 3.3, the two typical difference equations, are listed in Table III.

TABLE III

FINITE DIFFERENCE APPROXIMATION TO THE STRING'S DIFFERENTIAL EQUATION

TYPE	TYPICAL EQUATION	TOTAL TRUNCATION ERROR
CFD	$F(w_{s-1}-2w_s+w_{s+1})/h^2+q_s = 0$	$-\frac{h^2F}{12}\left(\frac{d^4w}{dx^4}\right)\tau_s$
HRMN	$F(w_{s-1}-2w_s+w_{s+1})/h^2+\frac{1}{12}(q_{s-1}+10q_s+q_{s+1})=0$	$\frac{h^4F}{240}\left(\frac{d^6w}{dx^6}\right)\tau_s$
CFD	Central Finite Difference	
HRMN	Hermitian Finite Difference	

IV. CONCLUSIONS

All of the equations derived give tridiagonal matrices centered about the principal diagonal. For a given number of nodes, a computer program using these equations should have approximately equal storage requirements, computation time and round-off error.

There are two ways to decrease the truncation error. One is to use higher order difference equations and the other is to reduce the interval, making more nodes. Normally, both of these cause greater round-off error, storage requirements, and computation time, since the first increases the number of non-zero terms in the matrix and the second increases the size of the matrix.

The higher order equations derived for this thesis do not increase the size of the displacement matrix but do change the load matrix from a diagonal to a tridiagonal matrix.

The object of any numerical analysis method is to obtain as accurate an answer as possible within the shortest computation time. Of the two finite difference solutions to the differential equation, the Hermitian method is the better of the two from the standpoint of truncation error.

For the equations derived from the energy solution, the truncation error is composed of three parts. Two parts are associated with the evaluation of the strain energy and

work integrals and are of $O(h^2)$ for the trapezoidal rule and $O(h^4)$ for Simpson's rule. The third is the one associated with the evaluation of the derivatives in the strain energy integral and is listed in Tables I and II. This part was of $O(h^2)$ when the central finite difference expression was used and of $O(h^4)$ when the higher order finite difference expression was used.

From the comparison of the equations made in this thesis by partial truncation error it would appear that the energy equations derived using Simpson's rule and the higher-order finite difference equations are the better equations. However, it must be stressed that this evaluation is based only on the apparent truncation error and is, therefore, not conclusive.

There is some difficulty in establishing the total truncation error associated with the equations derived from the energy approach. Consequently, it is recommended that these equations be evaluated on the computer so that a complete comparison can be made not only of truncation error but also of storage requirements and computation time.

APPENDIX A

THE HIGHER ORDER FINITE DIFFERENCE EQUATION FOR THE FIRST DERIVATIVE

Writing the Taylor series for w_{s+1} and taking the derivative twice gives

$$\left(\frac{d^2w}{dx^2}\right)_{s+1} = \left(\frac{d^2w}{dx^2}\right)_s + h\left(\frac{d^3w}{dx^3}\right)_s + \frac{h^2}{2}\left(\frac{d^4w}{dx^4}\right)_s + \frac{h^3}{6}\left(\frac{d^5w}{dx^5}\right)_s + \ldots \tag{A.1}$$

and for w_{s-1} gives

$$\left(\frac{d^2w}{dx^2}\right)_{s-1} = \left(\frac{d^2w}{dx^2}\right)_s - h\left(\frac{d^3w}{dx^3}\right)_s + \frac{h^2}{2}\left(\frac{d^4w}{dx^4}\right)_s - \frac{h^3}{6}\left(\frac{d^5w}{dx^5}\right)_s + \ldots \tag{A.2}$$

Subtracting Equation A.2 from Equation A.1 gives

$$\left(\frac{d^2w}{dx^2}\right)_{s+1} - \left(\frac{d^2w}{dx^2}\right)_{s-1} = 2h\left(\frac{d^3w}{dx^3}\right)_s + \frac{h^3}{3}\left(\frac{d^5w}{dx^5}\right)_s + \ldots \tag{A.3}$$

Solving this for $\left(\frac{d^3w}{dx^3}\right)_s$ gives

$$\left(\frac{d^3w}{dx^3}\right)_s = \frac{1}{2h}\left[\left(\frac{d^2w}{dx^2}\right)_{s+1} - \left(\frac{d^2w}{dx^2}\right)_{s-1}\right] - \frac{h^2}{6}\left(\frac{d^5w}{dx^5}\right)_s + \ldots \tag{A.4}$$

Substituting Equation A.4 into the central finite difference equation, Equation 2.4, for the non-staggered interval gives

$$\left(\frac{dw}{dx}\right)_s = \frac{1}{2h}(w_{s+1}-w_{s-1}) - \frac{h}{12}\left[\left(\frac{d^2w}{dx^2}\right)_{s+1} - \left(\frac{d^2w}{dx^2}\right)_{s-1}\right] + \frac{7h^4}{360}\left(\frac{d^5w}{dx^5}\right)_s + \cdots$$

(A.5)

The differential equation for the string is

$$\left(\frac{d^2w}{dx^2}\right)_s \equiv - \frac{q_s}{F} \qquad \text{(A.6)}$$

Substituting Equation A.6 into Equation A.5 and truncating gives

$$\left(\frac{dw}{dx}\right)_s = \frac{1}{2h}(w_{s+1}-w_{s-1}) + \frac{h}{12F}(q_{s+1}-q_{s-1}) + \frac{7h^4}{360}\left(\frac{d^5w}{dx^5}\right)_{\tau_s}$$

(A.7)

Equation A.7 is the higher order finite difference equation for the non-staggered scheme.

APPENDIX B

THE DERIVATION OF THE FORMULAS AND EQUATIONS USED WITH THE STAGGERED SCHEME

A. THE CENTRAL FINITE DIFFERENCE EQUATION FOR THE FIRST DERIVATIVE

The Taylor series for w_{s+1} is

$$w_{s+1} = w_{s+\frac{1}{2}} + \frac{h}{2}\left(\frac{dw}{dx}\right)_{s+\frac{1}{2}} + \frac{h^2}{8}\left(\frac{d^2w}{dx^2}\right)_{s+\frac{1}{2}} + \frac{h^3}{48}\left(\frac{d^3w}{dx^3}\right)_{s+\frac{1}{2}} + \frac{h^4}{384}\left(\frac{d^4w}{dx^4}\right)_{s+\frac{1}{2}} + \frac{h^5}{3840}\left(\frac{d^5w}{dx^5}\right)_{s+\frac{1}{2}} + \ldots \tag{B.1}$$

and Taylor series for w_s is

$$w_s = w_{s+\frac{1}{2}} - \frac{h}{2}\left(\frac{dw}{dx}\right)_{s+\frac{1}{2}} + \frac{h^2}{8}\left(\frac{d^2w}{dx^2}\right)_{s+\frac{1}{2}} - \frac{h^3}{48}\left(\frac{d^3w}{dx^3}\right)_{s+\frac{1}{2}} + \frac{h^4}{384}\left(\frac{d^4w}{dx^4}\right)_{s+\frac{1}{2}} - \frac{h^5}{3840}\left(\frac{d^5w}{dx^5}\right)_{s+\frac{1}{2}} + \ldots \tag{B.2}$$

Subtracting Equation B.2 from Equation B.1 gives

$$w_{s+1} - w_s = h\left(\frac{dw}{dx}\right)_{s+\frac{1}{2}} + \frac{h^3}{24}\left(\frac{d^3w}{dx^3}\right)_{s+\frac{1}{2}} + \frac{h^5}{1920}\left(\frac{d^5w}{dx^5}\right)_{s+\frac{1}{2}} + \ldots \tag{B.3}$$

Solving this for $\left(\frac{dw}{dx}\right)_{s+\frac{1}{2}}$ gives

$$\left(\frac{dw}{dx}\right)_{s+\frac{1}{2}} = \frac{1}{h}(w_{s+1}-w_s) - \frac{h^2}{24}\left(\frac{d^3w}{dx^3}\right)_{s+\frac{1}{2}} - \frac{h^4}{1920}\left(\frac{d^5w}{dx^5}\right)_{s+\frac{1}{2}} + \ldots \tag{B.4}$$

By truncating after the term in w the central finite difference equation for the first derivative for the staggered interval is

$$\left(\frac{dw}{dx}\right)_{s+\frac{1}{2}} = \frac{1}{h}(w_{s+1} - w_s) - \frac{h^2}{24}\left(\frac{d^3w}{dx^3}\right)_{\tau_{s+\frac{1}{2}}}$$

where $x_s \leq \tau_{s+\frac{1}{2}} \leq x_{s+1}$. (B.5)

B. THE HIGHER ORDER FINITE DIFFERENCE EQUATION FOR FIRST DERIVATIVE

Writing the Taylor series for w_{s+1} and taking the derivative twice gives

$$\left(\frac{d^2w}{dx^2}\right)_{s+1} = \left(\frac{d^2w}{dx^2}\right)_{s+\frac{1}{2}} + \frac{h}{2}\left(\frac{d^3w}{dx^3}\right)_{s+\frac{1}{2}} + \frac{h^2}{8}\left(\frac{d^4w}{dx^4}\right)_{s+\frac{1}{2}} + \frac{h^3}{48}\left(\frac{d^5w}{dx^5}\right)_{s+\frac{1}{2}} + \frac{h^4}{384}\left(\frac{d^6w}{dx^6}\right)_{s+\frac{1}{2}} + \ldots \tag{B.6}$$

and for w_s have

$$\left(\frac{d^2w}{dx^2}\right)_{s} = \left(\frac{d^2w}{dx^2}\right)_{s+\frac{1}{2}} - \frac{h}{2}\left(\frac{d^3w}{dx^3}\right)_{s+\frac{1}{2}} + \frac{h^2}{8}\left(\frac{d^4w}{dx^4}\right)_{s+\frac{1}{2}}$$

$$- \frac{h^3}{48}\left(\frac{d^5w}{dx^5}\right)_{s+\frac{1}{2}} + \frac{h^4}{384}\left(\frac{d^6w}{dx^6}\right)_{s+\frac{1}{2}} + \ldots \qquad \text{(B.7)}$$

Subtracting Equation B.7 from Equation B.6 gives

$$\left(\frac{d^2w}{dx^2}\right)_{s+1} - \left(\frac{d^2w}{dx^2}\right)_{s} = h\left(\frac{d^3w}{dx^3}\right)_{s+\frac{1}{2}} + \frac{h^3}{24}\left(\frac{d^5w}{dx^5}\right)_{s+\frac{1}{2}} + \ldots \qquad \text{(B.8)}$$

Solving for $\left(\frac{d^3w}{dx^3}\right)_{s+\frac{1}{2}}$ gives

$$\left(\frac{d^3w}{dx^3}\right)_{s+\frac{1}{2}} = \frac{1}{h}\left[\left(\frac{d^2w}{dx^2}\right)_{s+1} - \left(\frac{d^2w}{dx^2}\right)_{s}\right] - \frac{h^2}{24}\left(\frac{d^5w}{dx^5}\right)_{s+\frac{1}{2}} + \ldots \qquad \text{(B.9)}$$

Substituting Equation B.9 into Equation B.4 gives

$$\left(\frac{dw}{dx}\right)_{s+\frac{1}{2}} = \frac{1}{h}\left(w_{s+1} - w_s\right) - \frac{h}{24}\left[\left(\frac{d^2w}{dx^2}\right)_{s+1} - \left(\frac{d^2w}{dx^2}\right)_{s}\right]$$
$$+ \frac{7h^4}{5760}\left(\frac{d^5w}{dx^5}\right)_{s+\frac{1}{2}} + \ldots \qquad \text{(B.10)}$$

Truncating the series gives

$$\left(\frac{dw}{dx}\right)_{s+\frac{1}{2}} = \frac{1}{h}\left(w_{s+1} - w_s\right) - \frac{h}{24}\left[\left(\frac{d^2w}{dx^2}\right)_{s+1} - \left(\frac{d^2w}{dx^2}\right)_{s}\right]$$
$$+ \frac{7h^4}{5760}\left(\frac{d^5w}{dx^5}\right)_{\tau_{s+\frac{1}{2}}} \qquad \text{(B.11)}$$

The differential equation for the string is

$$\left(\frac{d^2w}{dx^2}\right)_s \equiv - \frac{q_s}{F}$$

Substituting this into Equation B. 11 gives

$$\left(\frac{dw}{dx}\right)_{s+\frac{1}{2}} = \frac{1}{h}(w_{s+1} - w_s) + \frac{h}{24F}(q_{s+1} - q_s)$$

$$+ \frac{7h^4}{5760}\left(\frac{d^5w}{dx^5}\right)_{\tau\ s+\frac{1}{2}} \qquad \text{(B.12)}$$

Equation B.12 is the higher order finite difference equation for the staggered scheme.

APPENDIX C

THE DERIVATION OF THE HERMITIAN FINITE DIFFERENCE EQUATION FOR THE SECOND DERIVATIVE

Adding Equations A.1 and A.2 gives

$$\left(\frac{d^2w}{dx^2}\right)_{s-1} - 2\left(\frac{d^2w}{dx^2}\right)_s + \left(\frac{d^2w}{dx^2}\right)_{s+1} = h^2 \left(\frac{d^4w}{dx^4}\right)_s$$

$$+ \frac{h^4}{12} \left(\frac{d^6w}{dx^6}\right)_s + \ldots \qquad \text{(C.1)}$$

Solving Equation C.1 for $\left(\frac{d^4w}{dx^4}\right)_s$ gives

$$\left(\frac{d^4w}{dx^4}\right)_s = \frac{1}{h^2}\left[\left(\frac{d^2w}{dx^2}\right)_{s-1} - 2\left(\frac{d^2w}{dx^2}\right)_s + \left(\frac{d^2w}{dx^2}\right)_{s+1}\right] - \frac{h^2}{12}\left(\frac{d^6w}{dx^6}\right)_s + \ldots$$

$$\text{(C.2)}$$

The Taylor series derived expression for the second derivative [1] is

$$\left(\frac{d^2w}{dx^2}\right)_s = \frac{1}{h^2}\left(w_{s-1} - 2w_s + w_{s+1}\right) - \frac{h^2}{12}\left(\frac{d^4w}{dx^4}\right)_s - \frac{h^4}{360}\left(\frac{d^6w}{dx^6}\right)_s + \ldots$$

$$\text{(C.3)}$$

Substituting Equation C.2 into Equation C.3 gives

$$\left(\frac{d^2w}{dx^2}\right)_s = \frac{1}{h^2}(w_{s-1}-2w_s+w_{s+1}) - \frac{1}{12}\left[\left(\frac{d^2w}{dx^2}\right)_{s-1} -2\left(\frac{d^2w}{dx^2}\right)_s + \left(\frac{d^2w}{dx^2}\right)_{s+1}\right] + \frac{h^4}{240}\left(\frac{d^6w}{dx^6}\right)_s + \cdots \tag{C.4}$$

Solving Equation C.4 for $(w_{s-1}-2w_s+w_{s+1})$ gives

$$(w_{s-1}-2w_s+w_{s+1})/h^2 = \frac{1}{12}\left[\left(\frac{d^2w}{dx^2}\right)_{s-1} + 10\left(\frac{d^2w}{dx^2}\right)_s + \left(\frac{d^2w}{dx^2}\right)_{s+1}\right] - \frac{h^4}{240}\left(\frac{d^6w}{dx^6}\right)\tau_s \tag{C.5}$$

Equation C.5 is the Hermitian equation for the second derivative.

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13. ABSTRACT

The principle of virtual work was used to determine several typical difference equations governing the displacement of a taut string subjected to a distributed transverse load. The integrals for the strain energy and the external work were approximated by both the trapezoidal rule and Simpson's rule. Further, the integrand in the strain energy expression was replaced by two different finite-difference approximations. For comparison, two typical difference equations were developed from the differential equation for the string using central finite differences and the Hermitian differencing scheme. The finite difference equations, all of which are tridiagonal, were evaluated by comparing truncation error.

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